Steady State Analysis of a Bulk Queueing Model with Multiple Vacations, Accessible Batches and Closedown Times

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Abstract--- In this paper, a generalized non-Markovian bulk arrival bulk service queueing system is studied considering multiple vacations and closedown times. The service starts only if minimum of ‘a’ customers are available in the queue. At the service completion epoch, if the number of customers is \( \xi \), where \( a \leq \xi \leq d - 1 \) \( (d \leq b) \) then the server takes the entire queue for batch service and admits the subsequent arrivals for service till the service of the current batch is over, or the accessible limit \( d \) is reached, whichever occurs first. At the service initiation epoch, if the number of customers waiting in the queue \( 1\xi' \) is at least ‘d’ \( (a \leq d \leq b) \), then the server takes \( \min(\xi, b) \) customers for service and does not allow further arrival into the batch. On completion of a service, if the queue length is less than ‘a’, then the server performs a closedown work such as, shutting down the machine, removing the tools etc. Following closedown work, the server leaves for a vacation of random length irrespective of queue length. When the server returns from a vacation and if the queue length is still less than ‘a’, he leaves for another vacation and so on until he finds at least ‘a’ customers waiting for service in the queue.

Keywords--- Bulk Queue, Multiple Vacations, Busy Period.  
Steady State Solution.

I. INTRODUCTION

MANY researchers have concentrated on bulk service queueing models, in which once the service is started arriving customers cannot enter the service station though enough space is available to accommodate them. It can be observed in many practical situations that arriving customers will be considered for service with current batch in service with some restriction (e.g. cinema hall). The concept of non-accessibility while receiving service, has been studied by Weiss [119], Sivasamy [109] analyzed a Markovian single arrival bulk service queue with accessible and non-accessible batches. Borthakur and Medhi (1974) have studied a queueing system with arrival and services in batches of variable size. Zhang (2006) considered a single server M/G/1 queueing system with server vacations of two types and two threshold policies. Sharma and Jain [104] obtained results for average queue length for state dependent Markovian single arrival bulk service queueing system with accessible and non-accessible batches. Krishna Reddy et.al (1998) have discussed on M/G(a,b)/1 model with N-policy multiple vacations and setup times. Wang et.al(2007) have analyzed a single unreliable server in an M\(^x\)/M/1 queueing system with multiple vacations. Sharma et al. established the expression for average queue length for state dependent M\(^x\)/M(a,d,b)/1 queue with accessible and non-accessible batches without vacations. In the literature, only less attention is given for general service because of the complexity in getting a closed form solution.

II. MATHEMATICAL MODEL

In this paper, a generalized non-Markovian bulk arrival bulk service queueing system is studied considering multiple vacations and closedown times. The service starts only if minimum of ‘a’ customers are available in the queue. At the service completion epoch, if the number of customers is \( \xi \), where \( a \leq \xi \leq d - 1 \) \( (d \leq b) \) then the server takes the entire queue for batch service and admits the subsequent arrivals for service till the service of the current batch is over, or the accessible limit \( d \) is reached, whichever occurs first. At the service initiation epoch, if the number of customers waiting in the queue \( 1\xi' \) is at least ‘d’ \( (a \leq d \leq b) \), then the server takes \( \min(\xi, b) \) customers for service and does not allow further arrival into the batch. On completion of a service, if the queue length is less than ‘a’, then the server performs a closedown work such as, shutting down the machine, removing the tools etc. Following closedown work, the server leaves for a vacation of random length irrespective of queue length. When the server returns from a vacation and if the queue length is still less than ‘a’, he leaves for another vacation and so on until he finds at least ‘a’ customers waiting for service in the queue.

The structure of the paper is as follows: the mathematical development of steady state equations is dealt in section 2. In section 3 queue size distribution is developed. Expressions for the expected length of idle period and busy period are derived in sections 4 and 5 respectively. Expected queue length for this queueing system is discussed in section 6. Expressions for mean waiting time is discussed in section 7. In section 8,9 and 10 cost model numerical illustrations and conclusions are presented.
be the probability that ‘k’ customers arrive in a batch at an arbitrary time epoch is obtained using supplementary variable technique. The complexity of general service accessible batch queueing system involving LST of unknown probability functions is overcome by proving a theorem using recursive approach. Expressions for expected queue length, expected length of idle period, expected length of busy period and expected waiting times are derived. A cost model of the queueing system is discussed. Numerical solution for particular values of parameters is presented.

Let X be the group size random variable, \( \lambda \) be the Poisson arrival rate, \( S(x) \), \( V(x) \) and \( C(x) \) denote the remaining service time of a batch, the remaining vacation time and closedown time, respectively. Let \( S_0(t), V_0(t) \) and \( C_0(t) \) be the density functions of service time, vacation time, and closedown time of a server at an arbitrary time epoch is obtained using supplementary variable technique. The complexity of general service accessible batch queueing system involving LST of unknown probability functions is overcome by proving a theorem using recursive approach. Expressions for expected queue length, expected length of idle period, expected length of busy period and expected waiting times are derived. A cost model of the queueing system is discussed. Numerical solution for particular values of parameters is presented.

For the proposed model, the probability generating function of the number of customers in the queue at an arbitrary time epoch is obtained using supplementary variable technique. The complexity of general service accessible batch queueing system involving inverse Laplace-Stieltjes transforms of \( s(x) \), \( v(x) \) and \( c(x) \) is overcome by proving a theorem using recursive approach. Expressions for expected queue length, expected length of idle period, expected length of busy period and expected waiting times are derived. A cost model of the queueing system is discussed. Numerical solution for particular values of parameters is presented.

Let \( S(t) \), \( V(t) \) and \( C(t) \) be the cumulative distributions of the service time, vacation time, and closedown time, respectively. Let \( s(x), v(x) \) and \( c(x) \) denote the Laplace-Stieltjes transforms of \( s(x) \), \( v(x) \) and \( c(x) \) respectively. The number of customers in the service and number of customers in the queue are denoted by \( N_S(t) \), \( N_Q(t) \) respectively.

The different states of the server at time ‘t’ are defined as follows:
- \( Y(t) = 0 \), if the server is busy with bulk service
- \( Y(t) = 1 \), if the server is doing closedown job
- \( Y(t) = 2 \), if the server is on vacation

and define \( Z(t) = j \), if the server is on \( j \)th vacation starting from the idle period.

To obtain system equations, the following probabilities are defined.

Let

\[
P_{ij}(x,t)dt = P\{N_S(t) = i, N_Q(t) = j \mid x \leq S(t) \leq x + dt, Y(t) = 0\}, \quad a \leq i \leq b, j \geq 0
\]

which means that there are \( i \) customers under service, \( j \) customers in the queue, and the server is busy with remaining service time of \( x \). In a similar manner, it is defined,

\[
P_{aj}(x,t)dt = P\{N_Q(t) = n, x \leq C(t) \leq x + dt, Y(t) = 1\}, \quad n \geq 0
\]

The following equations are obtained for the queueing system, using supplementary variable technique.

\[
P_{0i}(x - \Delta t, t + \Delta t) = P_{0i}(x,t)(1 - \lambda \Delta t) + \sum_{m=a}^{b} P_{mi}(0,t) s(x) \Delta t
\]

\[
+ \sum_{k=1}^{n} Q_{ik}(0,t) s(x) \Delta t + \sum_{k=1}^{n} Q_{i-k,d}(x,t) \lambda g_k \Delta t,
\]

\[
\quad a \leq i \leq d
\]

\[
P_{0j}(x - \Delta t, t + \Delta t) = P_{0j}(x,t)(1 - \lambda \Delta t) + \sum_{m=a}^{b} P_{mj}(0,t) s(x) \Delta t
\]

\[
+ \sum_{k=1}^{n} Q_{jk}(0,t) s(x) \Delta t,
\]

d + 1 \leq j \leq b

\[
P_{aj}(x,t) (a \leq j \leq d - 1) \text{ and } n \geq 1 \text{ is impossible.}
\]

\[
P_{dn}(x - \Delta t, t + \Delta t) = P_{dn}(x,t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} P_{d,n-k}(x,t) \lambda g_k \Delta t
\]

\[
+ \sum_{k=1}^{d} P_{d,k,0}(x,t) \lambda g_k \Delta t,
\]

\[
n \geq 1
\]

\[
P_{in}(x - \Delta t, t + \Delta t) = P_{in}(x,t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} P_{i,n-k}(x,t) \lambda g_k \Delta t,
\]

d < i < b - 1, \quad n \geq 1

\[
P_{pn}(x - \Delta t, t + \Delta t) = P_{pn}(x,t)(1 - \lambda \Delta t) + \sum_{m=a}^{b} P_{pm}(0,t) C(x) \Delta t
\]

\[
+ \sum_{k=1}^{n} P_{p,n-k}(x,t) \lambda g_k \Delta t,
\]

\[
n \geq 1
\]

\[
C_0(x - \Delta t, t + \Delta t) = C_0(x,t)(1 - \lambda \Delta t) + \sum_{m=a}^{b} P_{m,0}(0,t) C(x) \Delta t
\]

\[
+ \sum_{k=1}^{n} C_{n-k}(x,t) \lambda g_k \Delta t,
\]

\[
1 \leq n \leq a - 1
\]

\[
C_n(x - \Delta t, t + \Delta t) = C_n(x,t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} C_{n-k}(x,t) \lambda g_k \Delta t,
\]

\[
n \geq a
\]

\[
Q_{10}(x - \Delta t, t + \Delta t) = Q_{10}(x,t)(1 - \lambda \Delta t) + C_0(0,t) v(x) \Delta t
\]

\[
Q_{in}(x - \Delta t, t + \Delta t) = Q_{in}(x,t)(1 - \lambda \Delta t) + C_n(0,t) v(x) \Delta t
\]

\[
+ \sum_{k=1}^{n} Q_{i,n-k}(x,t) \lambda g_k \Delta t,
\]

\[
n \geq a, j \geq 2
\]

\[
Q_{ja}(x - \Delta t, t + \Delta t) = Q_{ja}(x,t)(1 - \lambda \Delta t) + Q_{j-1,a}(0,t) v(x) \Delta t
\]

\[
+ \sum_{k=1}^{n} Q_{j,n-k}(x,t) \lambda g_k \Delta t,
\]

\[
1 \leq n \leq a - 1, j \geq 2
\]

\[
Q_{ja}(x - \Delta t, t + \Delta t) = Q_{ja}(x,t)(1 - \lambda \Delta t) + \sum_{k=1}^{n} Q_{j,n-k}(x,t) \lambda g_k \Delta t,
\]

\[
n \geq a, j \geq 2
\]

From the above equations, the steady state queue size equations are obtained as follows:

\[
- \frac{d}{dx} P_{00}(x) = -\lambda P_{00}(x) + \sum_{m=a}^{b} P_{m,0}(0) s(x) + \sum_{k=1}^{n} Q_{0,k,0}(0) s(x) + \sum_{k=1}^{n} P_{d,k,0}(x) \lambda g_k,
\]

\[
a \leq x \leq d
\]
\[-\frac{d}{dx} P_{i0}(x) = -\lambda P_{i0}(x) + \sum_{n=d}^{b} P_{in}(0) s(x) + \sum_{j=i}^{n} Q_{ij}(0) s(x) ,\]
\[-\frac{d}{dx} P_{jm}(x) = -\lambda P_{jm}(x) + \sum_{n=d}^{b} P_{jn}(x) s(x) + \sum_{k=j}^{n} Q_{jk}(0) s(x) .\]

\[-\frac{d}{dx} C_{0}(x) = -\lambda C_{0}(x) + \sum_{m=a}^{b} P_{m0}(0) C(x) ,\]
\[-\frac{d}{dx} C_{n}(x) = -\lambda C_{n}(x) + \sum_{k=1}^{n} \lambda C_{k}(x) , \quad 1 \leq n \leq a - 1 .\]

\[-\frac{d}{dx} Q_{10}(x) = -\lambda Q_{10}(x) + C_{0}(0) v(x) ,\]
\[-\frac{d}{dx} Q_{jn}(x) = -\lambda Q_{jn}(x) + Q_{j-1,0}(0) v(x) .\]

\[-\frac{d}{dx} \tilde{P}_{i}(z) = \sum_{j=0}^{\infty} P_{j}(z) z^{j} ; \quad \tilde{P}_{i}(z) = \sum_{j=0}^{\infty} P_{j}(0) z^{j} , \quad d \leq i \leq b .\]
\[-\frac{d}{dx} \tilde{Q}_{j}(z) = \sum_{n=0}^{\infty} Q_{jn}(z) z^{n} ; \quad \tilde{Q}_{j}(z) = \sum_{n=0}^{\infty} Q_{jn}(0) z^{n} , \quad j \geq 1 .\]

\[-\frac{d}{dx} \tilde{C}(z) = \sum_{n=0}^{\infty} C_{n}(z) z^{n} ; \quad \tilde{C}(z) = \sum_{n=0}^{\infty} C_{n}(0) z^{n} .\]

To obtain the probability generating function of the queue size at an arbitrary time epoch, the following probability generating functions are defined:

\[-\theta \tilde{P}_{i}(z) - \tilde{P}_{i0}(z) = \lambda \tilde{P}_{i0}(z) + \sum_{n=d}^{b} \tilde{P}_{in}(0) \tilde{S}(z) - \sum_{j=i}^{n} \tilde{Q}_{ij}(0) \tilde{S}(z) ,\]
\[-\theta \tilde{P}_{i0}(z) - \tilde{P}_{i00}(z) = \lambda \tilde{P}_{i00}(z) - \sum_{n=d}^{b} \tilde{P}_{in}(0) \tilde{S}(z) - \sum_{j=i}^{n} \tilde{Q}_{ij}(0) \tilde{S}(z) ,\]
\[-\theta \tilde{P}_{in}(z) - \tilde{P}_{in0}(z) = \lambda \tilde{P}_{in0}(z) + \sum_{n=d}^{b} \tilde{P}_{jn}(z) \tilde{S}(z) - \sum_{k=n}^{\infty} \tilde{Q}_{nk}(0) \tilde{S}(z) ,\]
\[-\theta \tilde{P}_{in}^{-}(z) - \tilde{P}_{in0}^{-}(z) = \lambda \tilde{P}_{in0}^{-}(z) - \sum_{n=d}^{b} \tilde{P}_{jn}^{-}(z) \tilde{S}(z) - \sum_{k=n}^{\infty} \tilde{Q}_{nk}^{-}(0) \tilde{S}(z) ,\]
\[-\theta \tilde{Q}_{j0}(z) - \tilde{Q}_{j00}(z) = \lambda \tilde{Q}_{j00}(z) - \sum_{n=0}^{\infty} \tilde{Q}_{jn}(0) \tilde{V}(z) ,\]
\[-\theta \tilde{Q}_{jn}(z) - \tilde{Q}_{jn0}(z) = \lambda \tilde{Q}_{jn0}(z) - \sum_{n=0}^{\infty} \tilde{Q}_{jn}(0) \tilde{V}(z) .\]

III. QUEUE SIZE DISTRIBUTION

Multiplying the equation (22) by \(z^{a}\) and (23) by \(z^{b}(n \geq 1)\) taking summation from \(n=0\) to \(\infty\) and using (27), we get

\[-\theta \tilde{C}(z) - \tilde{C}(0) = \lambda \tilde{C}(0) - \sum_{k=1}^{a} \tilde{C}_{k}(0) \tilde{g}_{k} , \quad a \geq 1 .\]

Multiplying the equation (19) by \(z^{b}\), and taking summation from \(n=0\) to \(\infty\) and using (27), we get

\[-a \tilde{Q}_{i}(z) = \sum_{j=0}^{\infty} Q_{j}(z) z^{j} , \quad -a \tilde{Q}_{i}(z) = \sum_{j=0}^{\infty} Q_{j}(0) z^{j} .\]
\( (\theta - \lambda + \lambda X(z)) \hat{P}_t(z, \theta) = \hat{P}_t(z, 0) - S(\theta) \left[ \sum_{n=0}^{d} z^n m_0(0) + \sum_{i=0}^{\infty} Q_i(0) \right] \)

Multiplying (15) by \( z^n \) with \( i = b, (18) \) by \( z^n \) (\( n \geq 1 \)) and taking summation from \( n = 0 \) to \( \infty \) and using (27), we get.

\[ (\theta - \lambda + \lambda X(z)) \hat{P}_t(z, \theta) - \hat{P}_t(z, 0) = \theta \left[ \sum_{n=0}^{d} \sum_{i=0}^{b} z^n m_i(1) \right] \]

Substituting \( \theta = \lambda - \lambda X(z) \) in (28) through (33), we get

\[ \hat{Q}_1(z, 0) = \hat{V}(\lambda - \lambda X(z)) \hat{C}(z, 0) \] (34)

\[ \hat{Q}_j(z, 0) = \hat{V}(\lambda - \lambda X(z)) \sum_{n=0}^{d-1} \hat{Q}_{j-1, n}(0) z^n \] (35)

\[ \hat{C}(z, 0) = \hat{C}(\lambda - \lambda X(z)) \left( \sum_{m=a}^{d-1} \sum_{n=0}^{b} \frac{m_0(0) + \sum_{n=0}^{\infty} b}{m=m} \right) + \sum_{n=0}^{\infty} \sum_{i=0}^{b} \sum_{n=0}^{m} P_n(0) z^n \] (36)

\[ \hat{P}_t(z, 0) = \hat{S}(\lambda - \lambda X(z)) \left[ \sum_{m=a}^{d-1} \sum_{n=0}^{b} \sum_{n=0}^{m} P_n(0) z^n \right] + \sum_{n=0}^{\infty} \sum_{i=0}^{b} \sum_{n=0}^{m} Q_n(0) z^n \] (37)

\[ \hat{S}(\lambda - \lambda X(z)) \left[ \sum_{m=a}^{d-1} \sum_{n=0}^{b} P_n(0) + \sum_{n=0}^{\infty} b \right] = 0 \] (38)

\[ \hat{P}_b(z, 0) = \frac{z^b - S(\lambda - \lambda X(z))}{\theta - \lambda + \lambda X(z)} \] (40)

From the equations (28) and (34), we get

\[ \hat{Q}_1(z, \theta) = \hat{V}(\lambda - \lambda X(z)) \hat{C}(z, 0) \] (41)

From the equations (29) and (34) we get

\[ \hat{Q}_j(z, \theta) = \hat{V}(\lambda - \lambda X(z)) \hat{C}(z, 0) \] (42)

From the equations (30) and (36), it is got that

\[ \hat{C}(z, 0) = \frac{z^b - S(\lambda - \lambda X(z))}{\theta - \lambda + \lambda X(z)} \] (43)

From the equations (31) and (37), we get

\[ \hat{P}_t(z, \theta) = \frac{z^b - S(\lambda - \lambda X(z))}{\theta - \lambda + \lambda X(z)} \] (44)

From the equations (33) and (40), we get

\[ \hat{Q}_j(z, \theta) = \frac{z^b - S(\lambda - \lambda X(z))}{\theta - \lambda + \lambda X(z)} \] (45)

Let \( P(z) \) be the probability generating function of the queue size at an arbitrary time epoch. Then,

\[ P(z) = \sum_{i=0}^{d-1} \sum_{i=d}^{b} \hat{P}_i(z, \theta) + \sum_{i=0}^{b} \hat{P}_i(z, 0) + \hat{C}(z, 0) + \sum_{i=0}^{\infty} Q_i(0) \] (46)
are expressed in terms of known function \(S(\theta)\) and \(\phi(\theta)\), the above equation reduces to

\[
\frac{1}{-\lambda + \lambda X(z)} \left[ S(\lambda - \lambda X(z)) - 1 \right] + \sum_{i=d+1}^{b-1} S(\lambda - \lambda X(z)) c_m + \sum_{m=a}^{b-1} P_m(z(n)) z^n (X(z) - G_i(z))
\]

Let \(P_i = \sum_{m=d}^{b} P_{mi}(0)\), \(q_i = \sum_{l=1}^{\infty} Q_{li}(0)\), \(c_i = p_i + q_i\),

\[
\Psi(S,z) = \sum_{i=a}^{d-1} P_{i0}(0), \quad \phi(S,z) = \sum_{i=a}^{d-1} P_{i0}(\lambda - \lambda X(z) - P_{i0}(0))
\]

and

\[
f(S,z) = \frac{1}{-\lambda + \lambda X(z)} \left[ S(\lambda - \lambda X(z)) - 1 \right] + \sum_{m=a}^{b-1} P_m(z(n)) z^n (X(z) - G_i(z))
\]

Using equation (54), the equation (53) is simplified as

\[
P(z) = \left[ z^b - S(\lambda - \lambda X(z)) \right] \left[ -\lambda + \lambda X(z) \right] \Psi(S,z) + \lambda \left[ z^b - S(\lambda - \lambda X(z)) \right] \phi(S,z)
\]

\[
+ \left[ S(\lambda - \lambda X(z)) - 1 \right] f(S,z) + \left[ \sum_{n=0}^{a-1} P_n z^n \left( z^b - S(\lambda - \lambda X(z)) \right) \right] f(S,z) + \left[ \sum_{n=0}^{a-1} P_n z^n \left( z^b - S(\lambda - \lambda X(z)) \right) \right] f(S,z)
\]

In \(P(z)\) functions \(\Psi(S,z)\), \(\phi(S,z)\) and \(f(S,z)\) which involve LST of the unknown functions \(\tilde{P}_{i0}(0)\) are present.

While modeling a non-accessible batch service queue, such complexity will not occur. In order to resolve this complexity, \(\tilde{P}_{i0}(0)\) are expressed in terms of known function \(S(\theta)\), by proving the following theorem, using recursive approach.

**Theorem 1**

Let \(A_k\) be the set of positive integers whose sum is \(k\). The LST of the unknown probability function \(P_{a+n,0}(0)\), \(n = 1, 2, ..., d - 1 - a\), are expressed in terms of \(S(\theta)\) and \(S(n)\) higher derivatives \(S^{(n)}(\theta)\) at \(\theta = \lambda\), as

\[
P_{a+n,0} = \sum_{k=0}^{n} (-1)^{n(A_k)} \left( \prod_{i=A_k} \right) c_{a+n-k} T_n(A_k),
\]

where

\[
T_i = \frac{1}{\theta - \lambda} \sum_{i=1}^n \frac{S(\lambda - \lambda X(z)) - S(\lambda)}{\theta - \lambda}
\]

**Proof:**

Substituting \(i = a\) in equation (14), we get

\[
(\theta - \lambda) \tilde{P}_{a,0}(0) = P_{a,0}(0) - S(\theta) c_a
\]

when \(\theta = \lambda\), the above equation reduces to

\[
P_{a,0}(0) = S(\lambda) c_a
\]

Substituting (57) in (56), we get

\[
\tilde{P}_{a,0}(0) = \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a
\]

Substituting \(i + 1\) in (14), we get

\[
\tilde{P}_{a+n,0}(0) = \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a + \frac{P_{a,0}(0)\lambda g_1}{\theta - \lambda}
\]

\[
= T_0 c_a
\]

Substituting \(i = a + 1\) in (14), we get

\[
\tilde{P}_{a+n,0}(0) = \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a + \frac{P_{a,0}(0)\lambda g_1}{\theta - \lambda}
\]

\[
= \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a + \frac{P_{a,0}(0)\lambda g_1}{\theta - \lambda}
\]

\[
= T_0 c_a + \frac{S(\lambda) + T_0}{\theta - \lambda} c_a
\]

Substituting \(i = a + 2\) in (14) and using \(\tilde{P}_{a+1,0}(0)\) and \(\tilde{P}_{a,0}(0)\), we get
is given by

\[ P_{a+1,0}(\theta) = \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a + 2 - \lambda g_2 c_a \] \[ + \left( \frac{S(\lambda) - S(\theta)}{\theta - \lambda} - \lambda g_2 \right) \left( S(\lambda) - S(\theta) \right) \left( \frac{S(\lambda) - S(\theta)}{\theta - \lambda} \right) \frac{S(\lambda) - S(\theta)}{\theta - \lambda} \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a + 1 \]

\[ + \left( \frac{S(\lambda) - S(\theta)}{\theta - \lambda} - \lambda g_2 \right) \left( S(\lambda) - S(\theta) \right) \frac{S(\lambda) - S(\theta)}{\theta - \lambda} \frac{S(\lambda) - S(\theta)}{\theta - \lambda} \frac{S(\lambda) - S(\theta)}{\theta - \lambda} c_a + 1 \]

\[ = T_0 c_a + 2 - \lambda g_2 c_a T_1 - T_0 c_a T_1 + (\lambda g_1)(\lambda g_1)c_a T_2 \]

\[ = \sum_{k=0}^{\infty} (-1)^n(A_k) \left( \begin{array}{l} \prod \lambda g_i \\ i \in A_k \end{array} \right) c_{a+n-k} \]
Proof:

Using the equations (34) and (35) we get

\[ \sum_{l=1}^{\infty} Q_l(z) = \sum_{l=1}^{\infty} \left( C(z) \right)^l \]

\[ = \sum_{n=0}^{a-1} q_n z^n \]

\[ = \sum_{l=1}^{\infty} \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \alpha_i \beta_{n-i} \right) p_j z^n \]

Using the lemma 1 and lemma 2, we get

\[ \sum_{n=0}^{a-1} q_n z^n = \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} \alpha_i \beta_{n-i} \right) p_j z^n + \sum_{n=0}^{a-1} q_n z^n \]

Equating the coefficients of \( z^n \) on both sides of the above equation for \( n = 0, 1, 2, \ldots, a - 1 \), we have

\[ q_n = \sum_{j=0}^{n} \left( \sum_{i=0}^{n-j-1} \alpha_i \beta_{n-j-i} \right) p_j + \sum_{i=0}^{n-j-1} \alpha_i \beta_{n-j-i} q_i \]

Solving for \( q_n \), we get

\[ q_n = \frac{\sum_{j=0}^{n-j} \sum_{i=0}^{n-j} \alpha_i \beta_{n-j-i} p_j + \sum_{i=0}^{n-j} \alpha_i \beta_{n-j-i} q_i}{1 - \alpha_0} \]

Coefficient of \( P_n \) in \( q_n \) is

\[ \frac{\sum_{i=0}^{n-j} \alpha_i \beta_{n-j-i} k_0}{1 - \alpha_0} = k_0 \]

Coefficient of \( P_{n-1} \) in \( q_n \) is

\[ \frac{\sum_{i=0}^{n-j} \alpha_i \beta_{n-j-i} k_0}{1 - \alpha_0} = k_1 \]

where \( h_1 = \sum_{i=0}^{n-j} \alpha_i \beta_{n-j-i} k_0 \)

Coefficient of \( P_{n-2} \) in \( q_n \) is

\[ \frac{\sum_{i=0}^{n-j} \alpha_i \beta_{n-j-i} k_0}{1 - \alpha_0} = k_2 \]

Proceeding like this, we get

\[ \text{Coefficient of } P_0 \text{ in } q_n = \frac{h_n + \sum_{i=0}^{n} \alpha_i K_{n-i}}{1 - \alpha_0} = K_n \]

Therefore,

\[ q_n = \sum_{i=0}^{n} K_i p_{n-i} \]

Hence,

\[ \sum_{n=0}^{a-1} q_n z^n = \sum_{n=0}^{a-1} \left( \sum_{i=0}^{n} K_i p_{n-i} \right) z^n \]

\[ = \sum_{i=0}^{a-1} \left( \sum_{j=0}^{a-1} K_j z^{i+j} \right) p_j \]

Hence the theorem.

Equation (55) gives the probability generating function (PGF) of the queue size, but involves \( b \) unknowns \( C_0, C_1, C_2, \ldots, C_{b-1} \). To find these constants, Rouche’s theorem of complex variables can be used. By Rouche’s theorem, the expression \( z^b - S(\lambda - \lambda X(z)) \) has \( b - 1 \) zeros inside and 1 on the unit circle. Since \( P(z) \) is analytic within and on the unit circle, the numerator must vanish at these points which gives \( b \) equations in \( b \) unknowns. These equations can be solved by any numerical technique. Thus, the equation (55) gives the probability generating function of the number of customers in the queue at an arbitrary time.

Using the equation (55), it is derived that \( \rho < 1 \) as the steady state condition to be satisfied for the proposed model, where \( \rho = \lambda E(X) E(S) / b \).

**Particular Case**

When accessibility is not permitted (i.e. when \( d = a \)) then \( \psi(S, z) \) and \( \phi(S, z) \) in the equation (55) will become zero and on further simplification, we get

\[ P(z) = \frac{S(\lambda - \lambda X(z))}{S(\lambda - \lambda X(z)) - 1} \frac{1}{1 - \lambda X(z)} \]

The above \( P(z) \) gives PGF of the queue length distribution of \( M^\infty / G(a, b) / 1 \) queue system with multiple vacations without accessible batch service. The result exactly coincides with the queue length distribution of Arumuganathan and Jayakumar [6].

**IV. EXPECTED LENGTH OF IDLE PERIOD**

Expected idle period can be defined as the mean time gap between the completion epoch of a service and initiation epoch of the next service, inclusive of multiple vacations and closedown period. Let \( I \) be the random variable idle period. The expected length of idle period is given by

\[ E(I) = E(I_1) + E(C) \]

where \( I_1 \) is the random variable denoting idle period due to multiple vacation process and \( E(C) \) is the expected closedown time.

Let \( U \) be the random variable defined by

\[ U = 0 \text{, if the server finds at least 'a' customers after the first vacation} \]

\[ 1 \text{, if the server finds less than 'a' customers after the first vacation} \]

Then, using conditional expectation, the expected length of idle period \( E(I_1) \) is given by

\[ E(I_1) = E(I_1 / U = 0) P(U = 0) + E(I_1 / U = 1) P(U = 1) \]


\[ E(V) = E(U = 0) + (E(V) + E(I_1))P(U = 1) \]

where \( E(V) \) is the mean vacation time. Solving for \( E(I_1) \), we get

\[ E(I_1) = \frac{E(V)}{1 - P(U = 1)} = \frac{E(V)}{P(U = 0)} \quad (58) \]

To find \( P(U = 0) \), we do some algebra in the equations (27) and (34) and get

\[ Q_1(z,0) = \sum_{n=0}^{\infty} Q_{ln}(0) z^n \]

\[ = V(\ldots) \sum_{n=0}^{\infty} \alpha_n z^n \]

Equating the coefficients of \( z^n \) \((n = 0, 1, 2, \ldots, a - 1)\) on both sides, we get

\[ P(U = 0) = 1 - \sum_{n=0}^{a-1} Q_{ln}(0) \]

\[ = 1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n-1} \alpha_j \beta_{n-i-j} p_n \quad (59) \]

where \( \alpha_i, \beta_i \) are the probabilities of ‘i’ customers arrive during vacation and closedown time. Using (58) and (59), the expected idle period \( E(I) \) is obtained as

\[ E(I) = \frac{E(V)}{1 - \sum_{n=0}^{a-1} \sum_{i=0}^{n-1} \alpha_j \beta_{n-i-j}} \]

V. EXPECTED LENGTH OF BUSY PERIOD

In this section, the expected length of busy period which is useful to find the overall cost of the system is derived. Using the conditional expectation concept, the expected length of busy period is derived as follows:

Busy period is defined as the time interval from the moment when the server starts serving the queue, after returning from a vacation until the server leaves the system for another vacation.

Let \( B \) be the random variable for ‘busy period’. Define another random variable \( J \) as

\[ J = 0, \text{ if the server finds less than 'a' customers in the queue after first service} \]

\[ J = 1, \text{ if the server finds at least 'a' customers in the queue after first service} \]

Now the expected length of busy period \( E(B) \) is

\[ E(B) = E(B/J = 0) P(J = 0) + E(B/J = 1) P(J = 1) \]

\[ = E(B/J = 0) P(J = 0) + (E(B) + E(S)) P(J = 1) \]

\[ = E(B/J = 0) P(J = 0) + (E(B) + E(S)) (1 - P(J = 0)) \]

VI. EXPECTED QUEUE LENGTH

The expected queue length \( E(Q) \) (i.e., average number of customers waiting in the queue) at an arbitrary time epoch is obtained by differentiating \( p(z) \) at \( z = 1 \) and is given by

\[ E(Q) = \sum_{n=0}^{\infty} np_n = p_1 (1) \]

Using (58) and (59), the expected length of busy period \( E(B) \) is derived with the following assumptions.

\[ E(B) = \frac{E(S)}{P(J = 0)} = \frac{E(S)}{\sum_{i=0}^{a-1} p_i} \]

VII. EXPECTED WAITING TIME

The expected waiting time is obtained by using the Little’s formula as

\[ E(W) = \frac{E(Q)}{\lambda E(X)} \]

where \( E(S) \) is expected service time. Solving for \( E(B) \),

\[ E(B) = \frac{E(S)}{P(J = 0)} \]

\[ = \frac{E(S)}{\sum_{i=0}^{a-1} p_i} \]

VIII. COST MODEL

Cost analysis is an important phenomenon in any system. In this section, the total average cost of the queueing system is derived with the following assumptions.

\[ C_s : \text{Start-up cost per cycle} \]

\[ C_h : \text{Holding cost per customer per unit time} \]

\[ C_o : \text{Operating cost per unit time} \]

\[ C_r : \text{Reward due to vacation per unit time} \]
Cu : Closedown cost per unit time.

The length of cycle is the sum of the idle period and busy period. From the equations (59) and (60), the expected length of cycle, \( E(T_c) \) is obtained as

\[
E(T_c) = E(I) + E(B)
\]

(63)

The total average cost per unit is given by

\[
\text{Total average cost } = \text{start-up cost per cycle} + \text{holding cost of number of customers in the queue} + \text{operating cost } \rho^* + \text{closedown time cost} - \text{reward due to vacation per unit time.}
\]

(64)

where \( b \) is the service rate and \( \rho \) is the vacation time.

IX. NUMERICAL EXAMPLE

A numerical model is analysed with the following assumptions:

- Service time distribution is k-Erlang with \( k = 2 \) and service rate is \( \mu \).
- Batch size distribution of the arrival is geometric with mean = 2.
- Vacation time is exponential with parameter \( \alpha = 10 \).
- Closedown time is exponential with parameter \( \beta = 7 \).
- Minimum service capacity \( a = 3 \).
- Maximum service capacity \( b = 10 \).
- Accessible threshold value \( d = 4 \).
- Traffic intensity

\[
\rho = \frac{2\lambda k}{b\mu}
\]

Since \( k = 2 \) and \( b = 10 \), \( z^{12} - \widetilde{S}(\lambda - \lambda X(z)) \) will become a polynomial of degree 12 and it will have 9 roots inside, 2 roots outside and 1 on the unit circle \( |z| = 1 \). The zeros of the function \( z^{12} - \widetilde{S}(\lambda - \lambda X(z)) \) are found by using MATLAB [49] and using the same, the simultaneous equations are solved.

The expected queue length \( E(Q) \), expected length of idle period \( E(I) \), expected length of busy period \( E(B) \) and expected waiting time \( E(W) \) are computed and tabulated as detailed below:

Numerical results are presented in Table 1.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \rho )</th>
<th>( E(Q) )</th>
<th>( E(B) )</th>
<th>( E(I) )</th>
<th>( E(W) )</th>
<th>( ATC )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.4</td>
<td>2.2529</td>
<td>2.6676</td>
<td>0.1500</td>
<td>0.5632</td>
<td>4.5562</td>
</tr>
<tr>
<td>2.5</td>
<td>0.5</td>
<td>3.3378</td>
<td>2.9804</td>
<td>0.1417</td>
<td>0.6675</td>
<td>5.4590</td>
</tr>
<tr>
<td>3.0</td>
<td>0.6</td>
<td>4.9694</td>
<td>3.5834</td>
<td>0.1331</td>
<td>0.8282</td>
<td>6.5732</td>
</tr>
<tr>
<td>3.5</td>
<td>0.7</td>
<td>7.7042</td>
<td>4.6881</td>
<td>0.1256</td>
<td>1.1006</td>
<td>8.1941</td>
</tr>
<tr>
<td>4.0</td>
<td>0.8</td>
<td>13.2160</td>
<td>6.9908</td>
<td>0.1193</td>
<td>1.6520</td>
<td>11.1789</td>
</tr>
<tr>
<td>4.5</td>
<td>0.9</td>
<td>29.8744</td>
<td>14.0249</td>
<td>0.1142</td>
<td>3.3194</td>
<td>19.7247</td>
</tr>
</tbody>
</table>

In Table 1, the results of performance measures of the queuing system are presented for the service rate 2.0 and the arrival rate ranging from 2.0 to 4.5.

From the table (1), the following observations are made:

- Mean number of customers in the queue increases when the arrival rate increases.
- Expected length of idle period decreases, whereas that of busy period increases as arrival rate increases.
- Total average cost increases when arrival rate increases.

Thus the theoretical development of the model is justified with the numerical results, which are consistent with the fact that mean number of customers in the queue increases when the arrival rate increases.

X. CONCLUSION

An \( M^k /G(a,d,b)/1 \) queueing model with multiple vacations, accessible batch service and closedown times has been studied. The probability generating function of queue size at an arbitrary time is obtained using supplementary variable technique. The LST of unknown probability functions are
expressed in terms of known function $\tilde{S}(\theta)$, using recursive approach. Some important performance measures like expected queue size, expected length of idle and busy period are also obtained. A particular case of the model is also given. A cost model is developed and studied numerically.

**BIBLIOGRAPHIC NOTE**

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**REFERENCES**


